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The Multimodal Material Mind: Embodiment in mathematics education

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In a grade 5 class, the students are engaged in investigating the number of faces of a regular prism. The teacher has asked the following question: "If I know the name of a prism, can I deduce the number of its faces?" Addressing the class, Jim, one of the students, summarizes his findings and says:

Jim: Uh, yes, we can deduce the number of faces if we know the name of the prism because, if we take the example of a hexagonal prism, its bases are hexagons as the name says... [*He touches one of the bases of a plastic hexagonal prism that he is holding in his right hand; see Figure 26.1, Picture 1*].

Teacher: Excellent.

Jim: ... of the prism. So each edge [*touching one of the edges; see Figure 26.1, Picture 2*] has a face so a hexagon has 6 edges [*touching with his hands several faces of the hexagon; see Figure 26.1, Picture 3*]. So there are 6 lateral faces, and if we count the bases [*making a round gesture with the index of his right hand; see Figure 26.1, Picture 4*], it's 8 faces.

There are several elements in this short passage, which comes from a classroom in Ottawa, that have become relevant in contemporary discussions of mathematics education. These elements deal with a clear understanding of the roles played by the material geometric artifact, the tactile movement of Jim's hands around the physical geometric artifact and the linguistic activity that Jim deploys while touching it, and Jim's perception and imagination in the course of his embodied meaning-making process. Attention to these elements points to the idea that mathematical meanings that arise in teaching and learning are multimodal. More broadly, attention to these elements comes from new conceptions about human cognition, marked in particular by new understandings of the role of the body, language, and material culture. Distinct from traditional approaches, these conceptions highlight the cognitive role of semiotics and embodiment in mathematics thinking, teaching, and learning. Within these new conceptions, gestures, body posture, kinesthetic actions, artifacts, and signs in general are considered a fruitful array of resources to be taken into account when investigating how students learn and how teachers teach (e.g., Arzarello, 2006; Bautista & Roth, 2012; Borba & Villareal, 2006; Edwards, Radford, & Arzarello, 2009; Forest & Mercier, 2012; Radford & D'Amore, 2006; Radford, Schubring, & Seeger, 2008). These sensible and material resources are not considered mere epiphenomena of teaching and learning: They are conceptualized as central elements of the students' and teachers' mathematical thinking.

There are, however, a variety of interpretations of the role that humans' tactile-kinesthetic bodily experience of the world and their interaction with artifacts and material culture play in the way humans think and come [p. 701 starts here] to know.

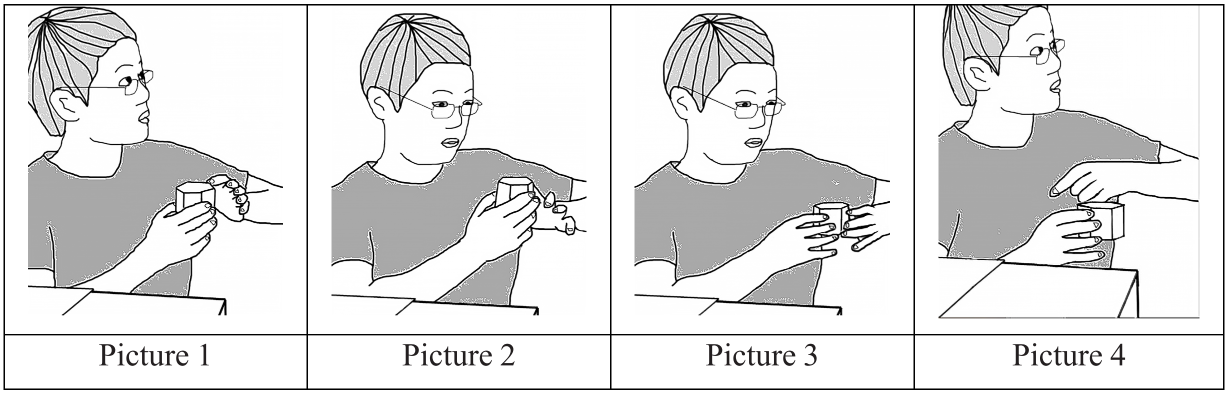


Figure 26.1. *Jim touching the plastic hexagon and making gestures.*

These interpretations depend on conceptions of cognition. For instance, some approaches inspired by cognitive linguistics (e.g., Fauconnier & Turner, 2002; Friedrich, 1970) emphasize the metaphoric dimension of language and the integrative constitution of embodied mental spaces (see, e.g., Edwards, 2009; Lakoff & Núñez, 2000; Yoon, Thomas, & Dreyfus, 2011). Other approaches, inspired by research in phenomenology, emphasize the "fleshy" nature of thought (Bautista & Roth, 2012; Roth, 2010; Thom & Roth, 2011), and yet others stress the materiality of cognition and its cultural-historical dimensions (de Freitas & Sinclair, 2013; Malafouris, 2012; Radford, 2013). All in all, these approaches claim that meaning and cognition are deeply rooted in physical, embodied existence and try to offer an answer to the questions of how meaning arises, and how thought is related to action, emotion, and perception.

This chapter provides a critical discussion of conceptualizations and applications of embodiment in mathematics education. In the next two sections, we discuss conceptions of human cognition and the role of embodiment and multimodality in general. This is followed by a section where we review several theories of embodiment in mathematics teaching and learning, including Piagetian inspired theory, semiotic oriented theory, enactivism, and phenomenological and materialist approaches. In the subsequent two sections, we delve in more detail into embodiment as featured in cultural-historical theory and cognitive linguistics, respectively. The chapter ends with a discussion of some open problems and possible new lines of inquiry.

The Human Mind

In the grade 5 example in the previous section, the students are exploring regular prisms. In the two previous lessons, the teacher and the students had discussed some key differences among three-dimensional solids; they had also distinguished between round solids and prisms. In this third lesson, the teacher shifts the investigation to the prisms' number of faces. The question is not asked for a specific prism. The question is asked *in general.* Like the other students in the class, Jim focuses on a specific prism; in his case, he chooses the hexagonal prism. He does know that the question is not about this particular prism. His answer attempts to cover other prisms as well. How can we interpret Jim's process of knowing?

Rationalist Epistemologies

Rationalist epistemologies of the Cartesian type argue that to cognize something, the human mind separates the thing to be known into parts. The human mind is supposed to operate analytically. It is, indeed, through the analysis of the thing's parts that the thing is finally known. The investigation of one or some of these parts (in this case, the consideration not of a prism in general, but a particular prism) would allow Jim to know proper- ties of the whole thing to be known—the prism *in general.* The 17th century Cartesian logician Antoine Arnauld (1861) calls this knowing process *abstraction:*

> The limited extent of our mind renders us incapable of comprehending perfectly things which are a little complex, in any other way than by considering them in their parts, and, as it were, through the phases which they are capable of receiving. This is what may be termed, generally, knowing by means of abstraction. (p. 45)

Within this context, Jim targets something general through something particular. Prompted by the teacher's [p. 702 starts here] question about prisms in general, he talks about the prism in general through a specific prism—the hexagonal prism, or rather, through a specific instance of the hexagonal prism, namely the one he holds in his hands.

This account of Jim's process of knowing goes back, in fact, to the Aristotelian concept of abstraction—a concept defined by the omission of attributes. The plastic nature of the hexagonal prism that Jim holds in his hand, its color, weight, and many other attributes, are omitted in order to think of the *hexagonal prism in general,* which in turn is apparently considered without its specific hexagonal property to think of the *prism in general.* In this account of knowing, the human mind is conceived of as equipped with the required discriminatory procedures that allow it to discard some attributes and to keep others. Additionally, objects of knowledge are conceived of as amenable to decomposition and analysis.

Empirical Epistemologies

Empirical epistemologies provide a different account of Jim's process of knowing. Thus, within a radical empirical epistemology—for instance, the one articulated by David Hume in *A Treatise of Human Nature* (1739/1965)— general properties of mathematical objects are not properties of the objects *per se.* They are properties that the individual, within the realm of her sensorial (also called sensible or sensuous) possibilities, bestows on objects. In this line of thought, the number of faces of a regular prism is not something that pertains to the prism *as such,* that is, to the prism as a kind of Platonic object independent of our senses. That number is rather the result of the sensorial experience that Jim makes of the various prisms he encounters in the classroom and in life. It is the impression of the sensorial experiences that Hume calls *ideas.* So, by touching, perceiving, holding, and moving his hands on the prism, Jim forms ideas. And in associating one idea with others, Jim forms more and more complex ideas. In a passage of *An Enquiry Concerning Human Understanding,* Hume (1748/1921) notes that although

> our thought seems to possess this unbounded liberty, we shall find, upon a nearer examination, that it is really confined within very narrow limits, and that all this creative power of the mind amounts to no more than the faculty of compounding, transposing, augmenting, or diminishing the materials afforded us by the senses and experience. (p. 16)

In this account, the number of faces of a prism is an *association of ideas* that originates in Jim's empirical experience of the world—what Hume (1748/1921) calls a "habit" of thought (p. 43).

The Epistemological Role of Embodiment and Multimodality in Rationalist Epistemologies

For the rationalist camp, cultural artifacts and the sentient body are not a source of knowledge. In his *Meditations,* Descartes (1641/1982) argues that things are grasped not through sensuous experiences but by the intellect alone: things "are not perceived because they are seen and touched, but only because they are correctly comprehended by the mind" (p. 26; our translation). In the same vein, another rationalist—Gottfried Wilhelm Leibniz—contended that "necessary truths such as found in pure mathematics, and particularly in arithmetic and in geometry, must have principles whose proof does not depend upon examples, nor consequently upon the testimony of the senses" (1705/1949, p. 44). A rationalist pedagogy would make little room for a sensuous experience in mathematics teaching and learning.

Embodiment in Empiricist Epistemologies

For the empiricist camp, by contrast, the sentient body is *the* source of knowledge. But because there is a limit to what humans can sense, the epistemic role of the body appears often as a constraint to what can be known. Hume (1739/1965) illustrated this point very well, arguing that the first principles of geometry, from which propositions are derived with alleged universality and exactness, rest, on closer examination, on "loose judgments of the senses and imagination" (pp. 70–71). This is why Hume (1739/1965) considered geometry an inexact science. Geometry's first principles are indeed still drawn from "the general appearance of the objects; and that appearance can never afford us any security, when we examine the prodigious minuteness of which nature is susceptible" (Hume, 1739/1965, p. 71). Because we cannot perceive minute angles, lines, and other geo- metric objects beyond human capacities, because we cannot transcend the threshold of human *perceptual minima,* "we have no standard…as to assure us of the truth of [geometric] proposition[s]" (1739/1965, p. 71). Regardless of the limits of what can be known as a result of the sensorial limits of our body, an empiricist pedagogy would nonetheless rely on, and encourage, sensorial experiences, since in this epistemological account we do not have any other source other than our body and our senses from which to learn and form ideas. [p. 703 starts here]

Embodiment in Kant's Epistemology

The previous discussion highlights the tremendous differences between the empiricist and the rationalist camps. The empiricists claim that nothing can be in the mind if it has not been in the senses first; the rationalists claim that nothing can be in the senses if it has not been in the mind first. Kant tried something daunting: to offer a theoretically coherent middle point between empiricism and rationalism—an empiricist-rationalist theory of knowing. This project is contained in *Critique of Pure Reason,* published in 1781 (Kant, 1781/2003).

In 1770 Kant was engaged in making a distinction between two kinds of knowledge: sensible and intelligible knowledge. Kant suggested that *sensible* knowledge involves all that can be known through our body and our senses, whereas *intelligible* knowledge comprises all that cannot be known by the senses but by the intellect or the mind only. In other words, sensible knowledge is what results from sensations, whereas intelligible knowledge is what results from representations of things that cannot by their own nature come before the senses. Kant was defending the idea that sensible and intelligible knowledge are two separate things. He went even further and claimed that there is no continuity between sensible (also called sensuous) and intelligible knowledge. These two realms of knowledge should be kept apart. He stated a methodological "precept": to take care not to allow the principles of sensuous cognition to transgress their limits and affect the intellectual concepts (Kant, 1770/1894). Presented with Jim's classroom episode dis- cussed previously, in 1770 Kant would have probably said that through the perceptual and tactile activity with the prism Jim came up with a *sensible concept* (i.e., a concept derived from experience), not with an *a priori concept* (in Kant's terminology, a concept independent of all experience and all impression of the senses). Kant would have warned us to follow his precept and to avoid confounding Jim's constructed experimental concept with the intelligible concept of the prism.

In *Critique of Pure Reason,* the sensible and the intelligible appear no longer as two separate realms but as related elements of human cognition. Presented with the same classroom episode in 1781, Kant would probably have said that Jim senses the physical prism through a very specific human capacity for being affected by material things. The physical prism appears to Jim not as such, directly, but as a kind of passive or receptive form of encountering the object mediated by the human modes of *sensibility:* sight, hearing, touch, taste, smell, and so forth. The effect on Jim that results from the receptive encounter of the object is what Kant calls *sensation.* Sensation, in other words, is the subjective act of being affected that results from the action of physical things on sensibility. It can be a specific color, sound, heat, and so forth. However, sensibility and sensation cannot lead by themselves to the concept of prism. They cannot yield knowledge of any object (Kant, 1781/2003, p. 73, A28/B44).

They are *alterations in Jim's body,* not the *qualities of the object* that make it a prism. How, then, can Jim come up with the concept of prism if sensuous experience is not enough? Kant introduces a crucial concept in this regard: the concept of *intuition,* which can roughly be translated as a passive form of representation through which an object (e.g., the prism) appears to Jim. Despite this refinement in the account of Jim's coming to know the prism and its properties, the question remains: How can Jim come up with the concept of prism if sensuous experience is not enough? Here is Kant's answer: The concept of prism does not arise out of what Jim can possibly discern through inspection of the physical artifact where he could read off its properties. The passive representation of the prism (i.e., the physical artifact) allows him "to bring out what was necessarily implied in the concept." (Kant, 1781/2003, p. 19, Bxii). In other words, if Jim was able to recognize the prism as such, it was not as a result of his embodied activity (which remains subjective and incapable of transcending the situatedness of his own experience), nor was it because of the manner in which the physical object appeared to Jim through his senses in its passive representation. The reason is that Jim mobilized a concept of prism that *he already had before any possible experience.* Indeed, in Kant's eyes, the concept of prism and all mathematical concepts carry with them their own conceptual proper- ties. These properties are universal and logically necessary. They do not depend on Jim or another individual. As a result, they cannot be derived from experience. Because for Kant mathematical concepts are not derived from experience, Kant called them *a priori:* "We shall understand by *a priori* knowledge, not knowledge independent of this or that experience, but knowledge absolutely independent of all experience" (1781/2003, p. 43, B3).

To sum up, the role of embodiment and multimodal- ity in Kant's theory of knowing was based on the distinction between two kinds of knowledge: the sensible and the intelligible. Although in the 1770 *Dissertation* these kinds of knowledge were conceived of as separate and different, in 1781 in *Critique of Pure Reason* they were conceived of as cooperating with each other (Kant, 1781/2003, p. 92, A50/B74). As a result, embodiment and [p.704 starts here] multimodality came to the forefront and gained a more central epistemological role. As Kant asserts in a famous passage in *Critique of Pure Reason,* without sensibility no intelligible object would be given to us. Without the intellect, no sensible object would be thought (Kant, 1781/2003, p. 93; A51/75 rephrased). Although embodiment and multimodality came to play a more prominent role in Kant's epistemological account of the senses, their contribution remained con- fined to providing the intellect with the raw material for it to be set into motion. In Kant's view, the empirical data becomes thinkable only because the intellect picks it up and endows it with conceptual content. For intelligible knowledge is *not* the content of generalized experience. This is why Jim, in the course of his mathematical experience, cannot possibly derive the universal and necessary properties of the prism that make it a prism in the mathematical sense. Universal necessity is unattainable from the apparent "necessity" that arises from experience, which remains always situated in space and time. If Jim comes to recognize the prism as an ideal object with its universal mathematical properties, it is not because of experience.

In Kant's account, the architectonic constitution of the human mind, with its arsenal of a priori knowledge and pure principles, provides Jim with "nothing but what may be called the pure schema of possible experience" (Kant, 1781/2003, p. 258, A236– 237/B295–296).

But what is a schema for Kant exactly? The schema is a kind of analogical procedure—a "monogram" (Kant, 1781/2003, p. 183, A 142/B 181)—that *unveils* the link between the intellectual and the sensual in the course of its empirical execution. In one respect, the schema must be intellectual; in another, it must be sensible. But the schema does not have to be confounded with an image:

> If five points be set alongside one another, thus, ... I have an image of the number five. But if, on the other hand, I think only a number in general, whether it be five or a hundred, this thought is rather the representation of a method whereby a multiplicity, for instance a thousand, may be represented in an image in conformity with a certain concept, than the image itself. For with such a number as a thousand the image can hardly be surveyed and compared with the concept. This representation of a universal procedure of imagination in providing an image for a concept, I entitle the schema of this concept. (Kant, 1781/2003, p. 182, A140/B179)

In saying that the schema is a method or universal procedure, Kant meant that its execution can be repeated again and again. The schema entails, in fact, a principle of iteration that links, thereby, knowledge and action (Radford, 2005).

Embodiment in Piaget's Epistemology

Considering himself a good Kantian, Piaget was not thrilled by Kant's *apriorism.* Piaget agreed with Kant that the object of reason is to inform experience. However, he did not accept Kant's idea that reason is something given a priori. The 28-year-old Piaget considered reason as constituted *in* experience: "Experience and reason are not two terms that we can isolate: Reason regulates experience and experience adapts reason." (Piaget, 1924, p. 587). But how could reason *emerge* from experience?

As we previously said, in Kant's theory of knowing, there always was an unbridgeable gap between the sensible and the intelligible, with the result that the latter cannot be thought of as a generalization or an abstraction of the former. Consequently, Kant did not need a concept of abstraction. Piaget, by contrast, was in need of one that would account for the emergence of reason *in* experience. He resorted to the aforementioned concept of *schema*, a concept that Piaget adapted to his needs by putting an emphasis on abstracting actions. Piaget (1970) argued that in abstraction there are two possibilities:

> The first is that, when we act upon an object, our knowledge is derived from the object itself. This is the point of view of empiricism in general, and it is valid in the case of experimental or empirical knowledge for the most part. But there is a second possibility: when we are acting upon an object, we can also take into account the action itself, or operation if you will.... In this hypothesis the abstraction is drawn not from the object that is acted upon, but from the action itself. It seems to me that this is the basis of logical and mathematical abstractions. (p. 16)

Piaget was eyeing the second kind of abstraction, which he termed *reflective abstraction.* Emancipated from its objects, reflective abstractions can be coordinated among themselves, for instance additively, temporally, and sequentially, giving rise to schemas that he interpreted as having "parallels in logical structures" (Piaget, 1970, p. 18) of a mathematical kind:

> Any given scheme in itself does not have a logical component, but schemes can be coordinated with one another, thus implying the general coordination of actions. These coordinations form a logic of actions that are the point of departure for the logical mathematical structures. (p. 42) [p. 705 starts here]

In Piaget's account, what allows Jim to recognize the prism as a mathematical object is the logical coordination of a series of schemas that he has constructed over the course of many years: figural schemas resulting from perceptual and tactile sensorimotor activities in the preschool years, followed by schemas of invariability of magnitudes, which are acquired when a child is around 9 or 10 years old and characterized by relations of de-centration vis-à-vis the objects, schemas of comparison of forms, going from intrafigural to interfigural relations, and so forth (see, e.g., Piaget, 1973). The logical coordination of those varied schemas culminates in the generation of projective and Euclidean geometric relations out of which the prism appears to Jim as it is: a mathematical prism with its universal properties.

Although Kant kept a separation of labor between the sensible and the intelligible, Piaget pleaded for a *developmental* relationship between the sensible and the intelligible. In Piaget's epistemology, sensorimotor actions give rise to a practical intelligence (characterized by an as-yet incomplete logical-mathematical structure) that expands itself into conceptual knowledge. Thus, with the arrival of the "semiotic function"—a function that has "the ability to represent something by a sign or a symbol or another object" (Piaget, 1970, p. 45)—actions and gestures become conceptual representations. Yet, the crucial point to note here is that, for Piaget, the individual's multimodal and embodied activity fades away. Intelligence becomes governed by logical-mathematical structures. The same is true of objects and artifacts. Reflective abstraction is abstraction emancipated from objects and artifacts. It converts actions into operations and operations into signs. But the allegedly structural nature of human thinking authorizes Piaget to remove artifacts, gestures, perception, and all embodied activity from the genetic analysis of the higher stages of intelligence. Piaget wrote:

> Reflective abstraction, which derives from the first concepts from the subject's actions, transforms the latter into operations, and these operations can sooner or later be carried out symbolically without any further attention being paid to the objects which were in any case "any whatever" from the start. (Beth & Piaget 1966, p. 238)

To sum up, Piaget emphasized the epistemological role of action and gesture. However, the emphasis on the operations' structure left little room for a thematization of the content of the operations and for a serious consideration of the semiotic systems and the cultural artifacts that children use. Thus, although in his experiments Piaget cleverly introduced a formidable series of ad hoc objects (blocks, fluid containers, trains, cars, objects of different weight and form), the object that the hand holds in the schema is unimportant. It may be "any whatever" from the start, as he says in the last quotation. Verillon and Rabardel comment that the object submitted to the Piagetian subject is fundamentally nonhistorical and nonsocial: "its main property is that it is structured by physical laws. . . . The introduction of artifacts in classic Piagetian experiments is mainly due to their convenience for highlighting the invariant properties of reality" (Verillon & Rabardel, 1995, p. 80).

Piaget's recourse to structuralism (even if it was a dynamic one) introduced irresoluble tensions in his epistemology—tensions that are proportional, we may say, to the ones Kant introduced in his by having recourse to apriorism. In Kant the tension appears between the sensible and the intelligible; in Piaget the tension appears between structure and object. What allows Jim to over- come the gap between his situated experience of the physical prism and the mathematical one with all its universal and necessary properties is the alleged "parallel- ism" between cognitive and mathematical structures. In both epistemologies, nevertheless, the common denominator is that the activity of the developed mind tends, in the end, to be largely confined to abstract mental activity (Radford, 2005).

In the following section we provide an overview of some perspectives on embodiment in mathematics education and examine how these approaches deal with the tensions identified above.

Embodiment in Mathematics Education

The Piagetian Legacy

Piaget's epistemology has had a significant impact on mathematics education and has also influenced the conception of embodiment. The influence is particularly patent in the so-called "process-object" theories; that is, theories that conceive of thinking as moving from the learner's actions to operation knowledge structures. Two examples are APOS theory (Dubinsky, 2002; Dubinsky & McDonald, 2001) and the "three worlds of mathematics" (Tall, 2013). APOS stands for actions, processes, objects, and schemas. The "three worlds of mathematics" refers to:

1. *conceptual embodiment,* which builds on perception and action to develop mental images that "become [p. 706 stars here] perfect mental entities" Tall, 2013, p. 16). For instance, "the number line develops in the embodied world from a physical line drawn with pencil and ruler to a 'perfect' platonic construction that has length but no thickness" (Tall, 2008, p. 14);
2. *operational symbolism,* which "grows out" of physical action into more or less flexible mathematical procedures; and
3. *axiomatic formalism,* which "builds formal knowledge in axiomatic systems specified by set-

theoretic definition" (Tall, 2013, p. 16).

One of the differences between APOS and the "three worlds of mathematics" perspective is the following. APOS theory focuses on the investigation of schema organization and genesis (Arnon et al., 2014); the "three worlds of mathematics" approach emphasizes the role of symbols and investigates the symbolic compression of processes according to whether the learner's attention is focused on objects, procedures, or symbols (Gray & Tall, 1994; Tall et al., 2001).

The "three worlds of mathematics" approach includes specific ideas about embodiment and mathematical thinking (de Lima & Tall, 2008; Tall, 2004, 2008, 2013; Tall & Mejia-Ramos, 2010; Watson & Tall, 2002). Thus, following a blend of the empiricist and rationalist philosophical traditions discussed in previous sections, Tall (2013) acknowledges that "mathematical thinking begins in human sensorimotor perception and action and is developed through language and symbolism" (p. 11). The meaning of the term embodiment in the "three worlds of mathematics" approach is explained as some- thing that is "consistent with the colloquial notion of 'giving a body' to an abstract idea" (Tall, 2004, p. 32).

Notice that the notion that ideas exist in an abstract, nonembodied form that can be "expressed" or "receive a body"—in mental imagery (or another type of representation)—is consistent with dualistic theories that separate the realm of ideas from the realm of the material and the sensible. This theoretical commitment, that philosopher of mathematics David Bostock (2009, p. 232) calls "conceptualism," has a clear implication for the manner in which methodological investigations are conducted. For instance, there is no need for an explicit analysis of the embodied cultural and conceptual sources of such things as symbols, mathematical definitions, or practices like proof. Instead, in the "three worlds of mathematics" approach these elements of thinking about and doing mathematics are analyzed from a taken-as-given world of mathematics and mathematicians. Real numbers, for example, are analyzed by contrasting their meaning and use within the three hierarchical worlds mentioned above: *embodied,* illustrated by a finger tracing "continuous motion" along a number line; *symbolic,* accompanied by "$\sqrt{2} = 1.4142\ldots$"; and *formal,* with a definition of a complete ordered field. Tall (2008) explains the hierarchical relationship as follows:

> Physically the number line can be traced with a finger and, as the finger passes from 1 to 2, it feels as if it goes through all the points in between. But when this is represented as decimals, each decimal expansion is a different point (except for the difficult case of recurring nines) and so it does not seem possible to imagine running through *all* the points between 1 and 2 in a finite time .... Formally, the real numbers R is an ordered field satisfying the completeness axiom. This involves entering a completely different world where addition is no longer defined by the algorithms of counting or decimal addition, instead it is simply asserted that for each pair of real numbers *a, b,* there is a third real number call[ed] the sum of *a* and *b* and denoted by *a+b.* (pp. 14–15)

To sum up, APOS theory offers a refined perspective to investigate the genesis of schemas, whereas the "three world of mathematics" approach offers a powerful framework to study the increasing transformation and compression of symbolism starting from an embodied level and moving toward flexible ways of using symbols and notations. Embodiment, nonetheless, remains a general category; the fate of embodied actions is to be superseded by flexible actions with symbols.

Multimodality

In other approaches to embodiment, the variety of embodied modalities to which students and teachers resort comes to the forefront. In particular, there is generally a stronger commitment to the essential role of the body even in abstract thought. These approaches may be termed *multimodal.* The term "multimodality" entered the mathematics education field after being borrowed from external research domains, ranging from neuroscience (for example, see Gallese & Lakoff, 2005) to communication studies (Kress, 2001, 2010). As Edwards and Robutti note (2014, p. 7), the "meanings used in these different fields of study are not mutually exclusive but intersect and complement each other." In mathematics education, the term multimodality is often used to underline both the relevance and mutual coexistence of a range of different cognitive, physical, and sensuous (e.g., perceptual, aural, tactile) modalities or resources [p. 707 starts here] playing a role in teaching-learning processes and, more broadly, in the production of mathematical meanings: "These resources or modalities include both oral and written symbolic communication as well as drawing, gesture, the manipulation of physical and electronic artifacts, and various kinds of bodily motion" (Radford, Edwards, & Arzarello, 2009, pp. 91–92).

An example of a multimodal approach is provided by the work of Abrahamson (2014), in what he terms "embodied design." This process involves the creation of physical tasks and computational environments that allow "proactive multimodal sensorimotor interaction" (Hutto, Kirchhoff, & Abrahamson, 2015, p. 375). For example, the Mathematical Imagery Trainer (MIT) allows the learner to engage kinesthetically and perceptually with the idea of proportionality. The student can change the color of a computer screen only when he holds one of his hands twice as far from the table as his other and maintains this ratio while moving his hands up or down. Thus, the introduction to proportionality is fully embodied in "non-symbolic perceptuomotor schemas" (Abrahamson, 2014, p. 1). Through the use of language, gesture, and, eventually, written inscriptions, the learner is assisted in reconciling his naïve, embodied, enacted experiences with more formal mathematical constructions.

How is embodiment understood here? What is the role of the students' gestures? Embodiment appears as a faculty of the body that has a constructive function in that it helps the creation of mathematical constructs in the course of learning (Alibali & Nathan, 2012; McNeill, 2000, 2005).

Semiotic Bundles

Another example of the multimodal approach has been developed by Arzarello and collaborators. Drawing on Vygotsky's work and neuroscience research, they stress the importance of the multimodal character of the students' semiotic activity in teaching and learning contexts. Here, the emphasis is not on schemas, as is the case of the Piagetian-influenced process-object theories mentioned before, but on the *evolution of signs* (Arzarello, 2006). In accordance with Vygotsky's early concept of sign, Arzarello refers to signs as mediating entities of thinking, much as tools are conceived of as mediating entities of labor. Within this context, gestures and other embodied resources to which students and teachers resort become signs, even if they do not present relatively formal rules of production as do language and algebraic and Cartesian graphic symbolism, through explicit grammatical or syntactic rules. In Arzarello's approach multimodality occurs through relationships between sets of signs (e.g., the set of speech language, the set of gestures, the set of algebraic symbols), produced and transformed according to their (formal or informal) nature and constituting a "semiotic bundle." A semiotic bundle is precisely formed by "i) A collection of semi- otic sets. ii) A set of relationships between the signs" (Arzarello, 2006, p. 281).

As we can see, the semiotic bundle considers the semiotic resources in a unifying manner, allowing for the description of learning through the evolution of signs as they are produced in the classroom by all participants. Arzarello, Paola, Robutti, and Sabena (2009) explain,

> Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher. (p. 100)

Due to the very general nature of the sign that is considered, the semiotic bundle includes the classical systems of signs (Ernest, 2006) or registers of represen- tation (Duval, 2006) as particular cases, but also gestures and embodied signs. Using the semiotic bundle, two kinds of interrelated analysis can be done: (1) a *synchronic analysis,* which focuses on the relationships between different signs in a certain moment, and (2) a *diachronic analysis,* which focuses on the evolution of signs (and the evolutions of the relationships between signs). Syn- chronic analysis allows for taking a kind of "picture" of the students' and teacher's mathematical activity from a semiotic point of view; diachronic analysis allows for obtaining a sort of multimodal semiotic "movie" of such an activity.

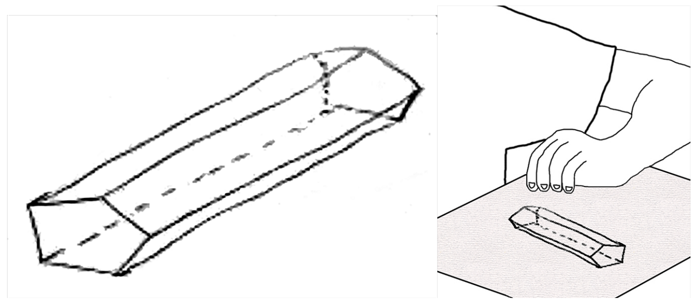
An example of a phenomenon detected with the synchronic view is the gesture-speech relationship in Jim's activity as described in the initial vignette. In this brief example, gestures and words cannot be considered separate because the meaning of one set completes the meaning of the other one (McNeill, 2000). As pointed out above, we can see for instance in line 3 of the transcript that gestures are co-timed with speech, and the sensuous aspects (touching, gazes) are deeply intermingled with speech to jointly express a reasoning: they co-live in the semiotic bundle.

The diachronic analysis is at the heart of the analysis carried out within the semiotic bundle perspective [p. 708 starts here] because it allows the researcher to determine whether and how an evolution of meanings occurred during the students' activity. For instance, going on with the discus- sion with the teacher and his mates, Jim draws on a sheet of paper a representation of a prism with a pentagonal base and says (Figure 26.2, Picture 1):

Jim: An edge, uh, um, every edge [*He slips his finger on the edge drawn on the sheet.*] is a side face [*Figure 26.2, Picture 2*] so it has five lateral faces.

So Jim produces a written diagram, in which we recognize (a variation of) the artifact he had interacted with, performs (a variation of) the gesture he had produced, and produces (a variation of) the reasoning he had done before. The variation consists both in the number of edges (5 for a pentagon and no longer 6 for a hexagon) and in the signs with which he interacts (a drawing and not the plastic artifact): the semiotic bundle has evolved. The changes are a possible hint of the generality with which Jim is reasoning, which, in turn, is embodied in the semiotic bundle relationships and evolution.

Looking at the evolution of the students' signs, the teacher can gain clues with respect to the students' understanding: the multimodal aspects of the activity can therefore help her decide whether or not to intervene in order to support the students. A didactic phenomenon reported in the literature is the so-called "semiotic game" (Arzarello et al., 2009), which happens when the teacher attunes to a certain semiotic set employed by the students (typically imitating a certain gesture) and couples it with another set (typically speech words or written mathematical symbols) to build a connection between personal and shared mathematical meanings. Therefore, semiotic games constitute an important strategy in the process of the appropriation of the culturally shared meaning of signs.



Picture 1 Picture 2

Figure 26.2. *Example of diachronic analysis: Diagram and gesture produced later by Jim.*

Enactivism

The origins of enactivism are in the biological roots of humans as described in the work of Maturana and Varela (1992) and in the phenomenological ideas of Merleau- Ponty(1945/1962). Enactivism shares with many embodied theories a critique of the Cartesian dualistic view of object and subject, the intellectual and the sensual, mind and body, and cognition and reality. It attempts to offer a middle view of the rationalist and subjectivist epistemological traditions that we outlined before. From an enactivist viewpoint, the world does not have pregiven properties that would exist independently from the human cognitive system. Nor can the cognitive system be thought of as projecting its own world—a world whose reality would be “a reflection of internal laws of the sys- tem” (Varela, Thompson, & Rosch, 1991, p. 172). Enactivists object to the objectivist rationalist view by arguing that cognitive categories are *experiential.* This is why, enactivists argue, the function of cognition is not to rep- resent the world: “cognition does not represent a world, it creates one” (Reid & Mgombelo, 2015, p. 176). At the same time, enactivists object to the subjectivist view by arguing that human cognitive categories belong to their shared biological and cultural world. Against the objectivist and subjectivist views they claim that the world out there and the individual “specify each other” (p. 172). “Our intention,” Varela, Thompson, and Rosch say, “is to bypass entirely this logical geography of inner versus outer by studying cognition not as recovery or projection but as *embodied action*” (p. 72). They explain the idea of embodied action as follows:

> By using the term *embodied* we mean to highlight two points: first, that cognition depends upon the kinds of experience that come from having a body with various sensorimotor capacities, and second, that these individual sensorimotor capacities are themselves embedded in a more encompassing biological, psychological, and cultural context. By using the term *action* we mean to emphasize once again that sensory and motor processes, perception and action, are fundamentally inseparable in lived cognition. Indeed, the two are not merely contingently linked in individuals; they have also evolved together. (Varela et al., 1991, pp. 172–173)

Within this perspective, knowing and perception are framed as active processes occurring directly through the interaction between the cognizing subject and the [p. 709 starts here] environment. Perception is determined by the structures of the perceiver, which are to be considered operation- ally closed and autonomous. Consequently, learning is conceptualized as a process of adaptation and restructuration caused by interactions within the environment, with learner and environment forming a complex dynamic system (a *structural coupling*). In their 1996 article "Cognition, Co-Emergence, Curriculum," Davis, Sumara, and Kieren speak of "the learner-in-her/his environment" and underline that in enactivist views, "context is not merely a place which contains the student; the student literally is part of the context" (quoted in Reid & Mgombelo, 2015, p. 177).

Phenomenological Approaches

Drawing on experimental and developmental psychology, cognitive science, and neuroscience, Nemirovsky and colleagues propose, like enactivism, a *nondualistic* embodied perspective on mathematical thinking and learning. One of the particularities of their approach is its phenomenological orientation and the important role ascribed to imagination and perceptuomotor integration in the learner's experience (Nemirovsky & Ferrara, 2009; Nemirovsky, Kelton, & Rhodehamel, 2013; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012). Perceptuomotor integration consists of a deep intertwining of perceptual and motor aspects of tool use or body movements.

Although in other phenomenological approaches (e.g., those drawing on Husserl's work) the primacy of senses is attributed to perception, in the radical embodied material phenomenology that Roth and colleagues propose, touch is the main sense. It is around touch that the sensations from the eyes and the other organs are coordinated, "especially with touch from the hands" (Roth, 2010, p. 11).

Roth discusses the example of Chris, a grade 2 student, comparing a material cube and a pizza box. The teacher asks what the pizza box would have to have to make it a cube. Prompted by the question, Chris moves [p.710 starts here] his hand along two edges of the pizza box. He utters the word "square" while pointing to the two sides again. Roth (2010) explains:

> With the movements and coordination of movements of eyes and hands, the world begins to emerge from touch. Chris's present experience is based on the coordination of hands with eyes, so that seeing the pizza box and moving the hand along one edge, then another edge, is but a realization of the coordination of hands and eyes and the concrete realization of the ability of moving them. (p. 11)

There are certain converging points between the radical embodied materialist phenomenology that Roth advocates and the enactivist approach previously described, but they do not coincide. A central difference is the immanent viewpoint that the materialist epistemology adopts: the existence of an original passivity that living organisms enjoy and which provides them with the possibility of affecting something and to be simultaneously affected—in the pizza box example, this immanence expresses itself in the preconceptual and preintentional movement of Chris's hand along the edge of the pizza box and, later on, a material cube: "It is the flesh, with its capacity of tact (i.e., sense of touch), con- tact (i.e., touched and being touched), and contingency that is the ground of all senses, sense-making efforts, and, therefore, knowledge" (Roth, 2010, p. 13)

But there is more to the difference between the radical

embodied materialist phenomenology and enactivism. Roth (2010) expresses it as follows:

> Varela et al. (1991) propose to look for knowledge at the "interface between body, society, and culture" (p. 179). In the position articulated here there is no interface: mind *is* in society and culture as much as society and culture are in the mind. Similar positions can be found in activity theory from L. S. Vygotsky via A. N. Leont'ev to the present day. Maturana and Varela (1980) take societies to be "systems of coupled human beings" (p. 118), whereas the position here is the converse: the specifically *human* being is a result of society rather than preceding coupling or, in activity theoretic terms, there is mind because there is society. (p. 16; emphasis in original)

Inclusive Materialism

De Freitas and Sinclair (2014) draw on the work of Barad (2007), Châtelet (2000), and Deleuze (1968/1994) to articulate an approach that they term *inclusive materialism*. Noting that theories of embodiment often remain focused on the individual learner and conceive of mathematical concepts as passive entities, they argue for a reconceptualization of the body that stretches conventional concepts. They suggest conceiving of the body as an assemblage "of human and non-human components" (2014, p. 25)—a heterogeneous assemblage of organic matter, concepts, tools, signs, diagrams, and objects (2014, p. 225).

This materialist ontological stance opens a space to talk of the human body as something that is more than what goes under the skin. It also makes room to talk of the *body of mathematics* and the *body of the tools* one uses in mathematical activity. "The new materialism we pro- pose," they say, "aims to embrace the 'body' of mathematics as that which forms an assemblage with the body of the mathematician, as well as the body of her tools/ symbols/diagrams" (de Freitas & Sinclair, 2013, p. 454).

The idea of the body that de Freitas and Sinclair pro- pose moves the discussion away from intentionality as one of the chief characteristics of individuals' actions and focuses on the field of agents and agency. As they note, "our aim here is to focus less on human intention and more on distributed agency. We want to problematize some of the ontological tenets underpinning particular conceptions of the human body as the principal administrator of its own participation" (2014, p. 19).

Although to be an agent and to be endowed with agency have usually been considered attributes of humans, in inclusive materialism, these attributes are not restricted to humans only. In this perspective, it makes sense to talk about matter as agentic entities. Thus, referring to Roth's (2010) analysis of the cube discussed above, de Freitas and Sinclair contend that "the matter of the cube and the matter of the mathematical concepts are also agents" (2014, p. 24).

Inclusive materialism stretches not only the concept of the body but also the concept of agency.

> The concept of agency must be rethought, because inclusive materialism problematizes the premise that any one part of the assemblage is the source of action, intention or will. Such problematizing will mean revising notions such as student agency, as well as advocacy or interventions for improving or sup- porting student agency. We will need to reconceive agency as operating within the relations of an ever-changing assemblage. (de Freitas & Sinclair, 2014, p. 33)

And this is precisely what Roth's (2010) analysis of the cube discussed above does not address adequately. Indeed, de Freitas and Sinclair (2014) argue that Roth's analysis fails to notice that the cube is not an inert object but rather an animated entity in "intra-action" with the [p. 711 starts here] student and the mathematical concept in a process of becoming:

> While Roth's cube example sheds light on the role of the body in learning, the analysis fails to do justice to the materiality of either the cube or the mathematics; that is to say, it fails to reckon with the way in which the cube is itself becoming-cube through its encounter with the child, shifting its own boundaries in this process of becoming. Roth treats the nonhuman material in this encounter as passive and inert. . . . Moreover, the mathematical concept of cube remains untouched and untroubled by the encounter, as though it were indeed an immaterial and inflexible concept that happens to be some- how manifest in this particular instance. (pp. 23–24)

In general, what de Freitas and Sinclair do not see in the current literature on the material aspects of mathematics is "how mathematical concepts partake of the material in operative, agential ways" (p. 40).

They locate the "forceful, animate, mobile, alive and material" (de Freitas & Sinclair, 2014, p. 226) nature of mathematical concepts in a conceptual category called the virtual. To understand the meaning of the virtual and virtuality, we need to go back to Deleuze's concept of the virtual. Deleuze (1968/1994, p. 209) contended that "the virtual must be defined as strictly a part of the real object—as though the object had one part of itself in the virtual into which it is plunged as though into an objective dimension." This is why "Every object is double without it being the case that the two halves resemble one another, one being a virtual image and the other an actual image" (p. 209). Inclusive materialism expands this idea to mathematical objects as well. As a result, "mathematics cannot be divorced from 'sensible matter,' and it is the virtual dimension of this matter that animates the mathematical concept. Mathematical entities are thus material objects with *virtual* and *actual* dimensions" (de Freitas & Sinclair, 2014, pp. 201–202; emphasis in original).

An object (mathematical or other) is hence a double object, made up of an actual image and a virtual one, and it is in the virtual one that we find the mobility of the concept. Considered from this posthumanist account, traditional teaching of mathematics does not attend to the virtual; it focuses on the logical. Now, the virtual, as considered in this approach, can be summoned or invoked. The virtual is something that can be "provoked," "recovered," "unleashed," and "conjured," but also "massacred" (de Freitas & Sinclair, 2014, p. 213). Gestures, diagrams, and mathematical notations are considered as "invoking a dynamic process of excavation that conjures the virtual in sensible matter" (2014, p. 67).

In the rest of the chapter we discuss two other approaches to embodiment in mathematics education. The first one comes from cultural-historical cultural theory and its dialectical materialist philosophy and the second one from cognitive linguistics.

Dialectical Materialism

Like some of the embodiment approaches discussed previously, dialectical materialism (Ilyenkov, 1982; Lefebvre, 2009) emphasizes the role of the body, matter, and the material world in knowing and becoming. Yet, material objects (e.g., the cube in Roth's, 2010, example mentioned above or the hexagonal prism alluded to in our introduction) are not conceived of as agentic entities. But neither are they considered the mere stuff that we touch with our hands, hear with our ears, or perceive with our eyes. They are considered bearers of sedimented human labor. That is, they are bearers of human intelligence and specific historical forms of human production that affect, in a definite way, the manner in which we come to know about the world. Thus, from the viewpoint of dialectical materialism, the cube that Chris holds in his hand in Roth's (2010) example and the hexagon that Jim holds in his (see Figure 26.1 in the introduction) are not conceptually neutral. These objects are bearers of a historical intelligence that has been produced and refined in the course of cultural development, providing the students with *potential* conceptual geometric categories through which they sort out and make sense of the world. The made-in-China plastic hexagon that Jim holds in his hand already intimates a way of seeing the world. It is a cultural artifact already embedded in a particular historical form of industry in a society of massive schooling and specific conveyed forms of knowing that are substantially different from those of Ancient Greece, the Middle Ages, or the pre-Columbian Maya cultures, for instance.

Dialectical materialism offers a conception of knowledge and the knowing subject as cultural, historical entities entangled in, and emerging from, material human activity (Leont'ev, 1978; Mikhailov, 1980). Within this perspective, the human subject is not a mere body. The human subject is the unique *individuation* of an ensemble of culturally and historically constituted ethical, social, political, and economic *relations.* As unique individuation of societal relations, the human subject is rather an entity in perpetual becoming—an unfolding and endless social, cultural, historical, material, and ideal (i.e., non-material) *project* of life. The human subject is something always resisting the identity with itself, I≠I.
[p. 712 starts here]

Radford (2009b, 2013, 2014b) has explored this line of theorizing the human subject to revisit embodiment and to consider cognition, sense, sensation, and matter in a new light. Cognition is conceptualized simultaneously as conceptual, embodied, *and* material. As a result, cognition is not seen through "conceptualist lenses"— that is, as something about *ideas* occurring *in* the head (Bostock 2009; Stevens, 2012). Cognition, the body, sense, sensation, and matter are considered kinds of a historical nature *intertwined* with each other. This theoretical approach, which Radford (2014b) terms "sensuous cognition," rests on a specific historical understanding of sense, sensation, materiality, and the conceptual realm. Within this theoretical perspective, our cognitive domain can only be understood as a culturally and historically constituted sentient form of creatively responding, acting, feeling, imagining, transforming, and making sense of the world. (p. 350)

As a result, the human senses are not conceived of as merely part of our phylogenetic evolved biological equipment. Perception, for instance, instead of being considered a sensorial synthesis carried out by a contemplative Cartesian subject (de Freitas, 2016) is considered a "highly evolved and specific mode of human action or praxis … [whose] characterization as only biological or physiological.. .is inadequate" (Wartofsky, 1979,

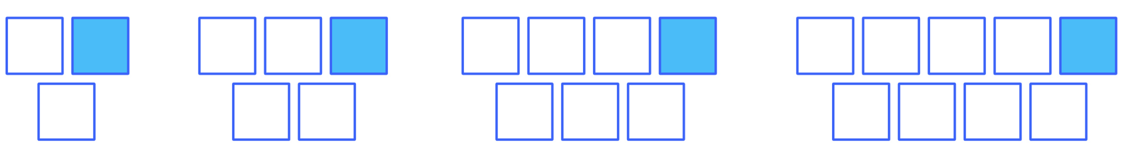
p. 189). Within the sensuous cognition approach, the biological orienting-adjusting reactions with which we are born undergo cultural transformation. Our sensorial- perceptual organs are converted into historically constituted complex forms of sensing (e.g., ways of seeing, touching, hearing, and tasting), leading to specific forms of human development (Radford, 2014b). The cultural transformation of the senses and their role in knowing can only be understood in the context of the "insertion of the individuals in that specific region of the world that is society, that is to say the set of connections by which the individuals come to exist with one another and with the world" (Fischbach, 2014, p. 8; our translation). In dialectic materialism, the name of this dynamic continuously unfolding and changing process of insertion of individuals in society is *activity*—material joint-activity.

To illustrate these ideas Radford (2014a) discusses an example that comes from a regular grade 2 class of 7- to 8-year-old students. In this example, the students worked on the sequence shown in Figure 26.3.

The students were invited to draw Terms 5 and 6. In subsequent questions they were invited to find out the number of squares in remote terms, such as Terms 12 and 25.

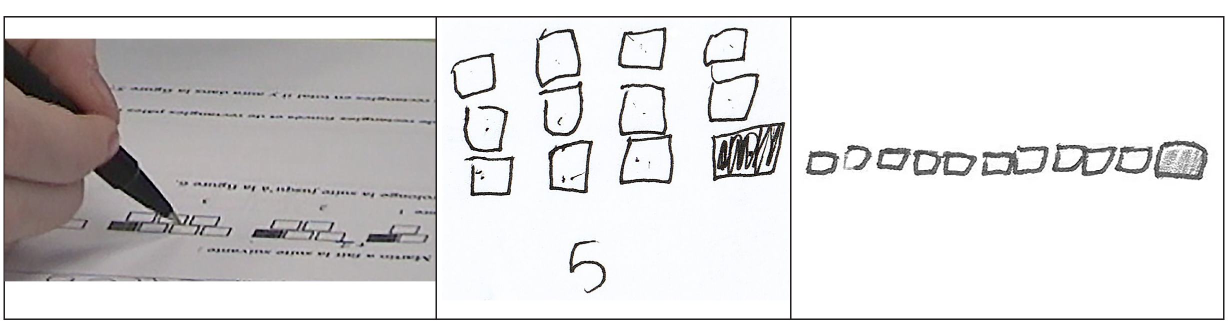
Radford notes that mathematicians often tend to *see* the terms of the sequence as made up of two rows. Then they scan the rows for *functional* clues between the number of the term and the number of squares on one row and the other. Mathematicians quickly realize that there are as many squares on the bottom row as the number of the term, and that there is one more square on the top row than the number of the term. They conclude that the general formula is $y = n + n + 1$, that is, $y = 2n + 1$. Or they notice the recursive relationship $T_n \square 1 = T_n + 2$ (an arithmetic sequence where the repeated addition is transformed into a multiplication). All this happens so quickly that it seems that the two rows and the recursive relationship stare us "in the face," to borrow an expression from Wartofsky (1968, p. 420). Yet, as Figure 26.4 intimates, things do not necessarily go that way for young students. Figure 26.4 shows two paradigmatic answers provided by two students: Carlos and James.

Figure 26.4, middle, shows in the interior of the squares the points left by the hand-pen counting device. These points are traces of Carlos's counting, which was also supported by uttered number names and a sustained perceptual activity. Against conceptualist trends, Radford (2009b) argues that "[T]hinking [is] not occur[ring] solely *in* the head but also *in* and *through* a sophisticated semiotic coordination of speech, body, gestures, symbols and tools" (p. 111). [p. 713 starts here]



Term 1 Term 2 Term 3 Term 4

Figure 26.3. *The first terms of a sequence that grade 2 students investigated in an algebra lesson. From "The Progressive Development of Early Embodied Algebraic*



Thinking" by L. Radford, 2014, Mathematics Education Research Journal*, 26(2), p. 262.*

Figure 26.4. *Left, Carlos, counting aloud, points sequentially to the squares in the top row of Term 3. Middle, Carlos's drawing of Term 5. Right: James's drawing of Term 5. From "The Progressive Development of Early Embodied Algebraic Thinking" by L. Radford, 2014,* Mathematics Education Research Journal*, 26(2), p. 263.*

In the sensuous cognition approach, the students' answers shown in Figure 26.4 are not assumed to mean that the students do not seek help in the *spatial configuration* of the terms of the sequence. As shown in Figure 26.4, left, Carlos meticulously points to the squares in the top row, one after the other in an orderly manner, and, once he finishes counting them, he starts counting the squares in the bottom row. However, the spatial configuration of the terms appears as an aid to perform a consecutive counting process only. The leading activity is centered on *numerosity*.

There is nothing wrong, of course, with the consecutive counting approach, except perhaps that it may reveal itself as very limiting to imagine and investigate remote terms, as it turned out to be in the case of this lesson. The students *see* the terms, but they do not necessarily see them as made up of two rows. To see the terms as made up of two rows already requires a kind of *theoretical seeing,* an algebraic way of perceiving. The eye has yet to be transformed from an organ of quotidian perception into a theoretician (Radford, 2010). This is why in the sensuous cognition approach, coming to know in a culturally and historically constituted form of knowing goes hand in hand with a transformation of our senses. We think (practically and theoretically) with and through our senses. This is why, within the sensuous cognition approach, perception, tactility, gestures, sounds, movement, and material objects do not mediate thinking. They are part of it.

Radford (2010) documents a key moment in the trans- formation of the senses of the grade 2 students mentioned above. Working with a small group of three students, and after the students had drawn Terms 5 and 6 in ways similar to those shown in Figure 26.5, the teacher engaged the students in an exploration of the patterns in which the rows come to the fore (see Figure 26.5).

The teacher says, "We will just look at the squares that are on the bottom." At the same time, to visually emphasize the object of attention and intention, the teacher makes three consecutive sliding gestures, each one going from the bottom row of Term 1 to the bot- tom row of Term 4. Figure 26.5, left, shows the beginning of the first sliding gesture. The teacher continues: "Only the ones on the bottom. Not the ones that are on the top. In Term 1 [she points with her two index fingers to the bottom row of Term 1; see Figure 26.5, right] how many [squares] are there?" Pointing, one of the students answers: "one." The teacher and the students continue rhythmically exploring the bottom row of Terms 2, 3, and 4, and also, through gestures and words, the non- perceptually accessible Terms 5, 6, 7, and 8. Then, they turn to the top row. Imagining the nonperceptually accessible terms is a fully sensuous process out of which an algebraic sense of the relations between the number of the terms and the number of squares in their bottom and top rows starts to emerge. Through the complex coordination of gestures, words, figures, and rhythm, [p. 714 starts here] the students start noticing a culturally and historically constituted theoretical way of seeing and gesturing.

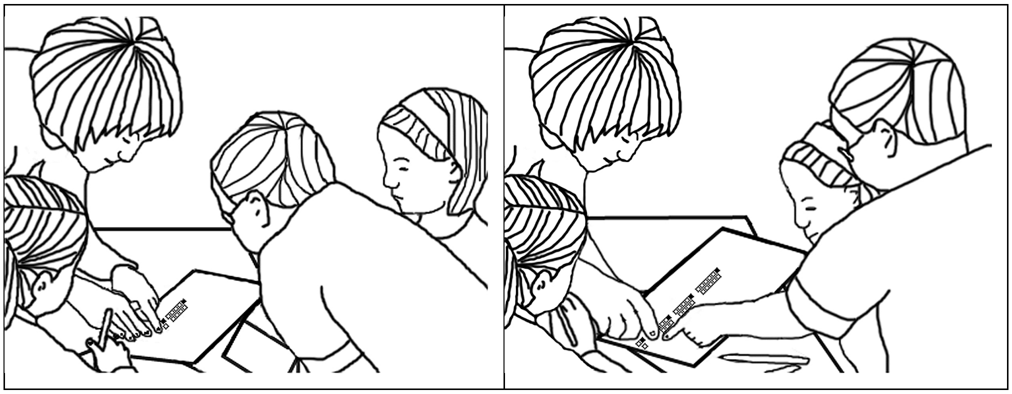


Figure 26.5. *Left, the teacher pointing to the bottom rows. Right, the students and the teacher counting together. From "Perceiving With the Eyes and With the Hands" by L. Radford, 2013,* RIPEM*,* 3*(1), p. 64.*

The students start discerning a new way of perceiving out of which an algebraic numerical-spatial structure becomes *apparent* and can be now applied to other terms of the sequence that are not in the students' perceptual field.

Within the sensuous cognition theoretical approach, the segment of the material and sensuous classroom joint activity out of which a mathematical form of thinking (in this example, an algebraic form of thinking about sequences) progressively appears in sensible conscious- ness is called a *semiotic node* (Radford, 2009a; Radford, Demers, Guzmán, & Cerulli, 2003). A semiotic node is not a set of signs. It is a *segment of joint activity* that usually includes a complex coordination of various sensorial and semiotic registers that the students and teachers mobilize in order to notice something (e.g., a mathematical structure or a mathematical concept at work). In the previous example, when the students are counting along with the teacher, the semiotic node includes the signs on the activity sheet, the teacher's sequence of gestures, the words that the teacher and the students pronounce simultaneously, the coordinated perception of the teacher and the students, the corporeal position of the students and the teacher, and rhythm as an encompassing sign that links gestures, perception, speech, and symbols.

The next day, the emerging awareness of the algebraic structure leads the students to suggest that the number of squares in Term 12 is "12 plus 12, plus 1." The structure of the semiotic node has been transformed: although rhythm still appears there in the prosodic flow of the utterance, it appears in a shorter and more direct manner. Also, the spatial deictic terms, such as "top" and "bottom," have disappeared, as have the pointing gestures. Radford (2008) calls this phenomenon a *semiotic contraction,* which results from making a choice between what counts as relevant and irrelevant, what needs to be said and not said, leading "to a contraction of expression" that is a material token of a "deeper level of conscious- ness and intelligibility" (2008, p. 90).

The concept of semiotic node is consistent with the idea of thinking featured in the sensuous cognition approach. Thinking is indeed considered to be made up of material and ideational components including (inner and outer) speech, sensuous forms of imagination, gestures, tactility, and actual actions with cultural artifacts. Now, conceiving of thinking as a sensuous, material process that resorts to the body and material culture does not mean that thinking is a collection of items. Rather, thinking is a dynamic *unity* of material and ideal components (Rieber & Carton, 1987). Thinking is something moving and unfolding—a movement of multiple corporeal, linguistic, symbolic, gestural, tactile, perceptual, physical, aesthetic, and emotional tonalities and positions.

Within the sensuous cognition approach, the investigation of semiotic nodes and their semiotic contraction is a crucial point in understanding teaching-learning processes. From a methodological viewpoint, the problem is to understand how, in classroom activity, the diverse sensorial modalities, the semiotic signs (linguistic, writ- ten symbols, diagrams, etc.), and cultural artifacts are *related, coordinated,* and *subsumed* into a new sensuous dynamic *unity* (Radford, 2012).

In the next section, we turn to a different conception of embodiment that comes from a contemporary approach to meaning making—cognitive linguistics—that has had a significant impact on conceptions of embodiment within mathematics education.

Cognitive Linguistics

The discipline of cognitive linguistics is based on the theory of embodied cognition, which, like enactivism, holds that the shared experience of existing as biological organisms who are born and grow up in a specific physical (and cultural) world provides the foundation for human language, thought, and meaning (Gibbs, 2006; Johnson, 1987, 2007; Lakoff & Johnson, 1999; Varela et al., 1991). More specifically, proponents of cognitive linguistics suggest that the relationship between elements of language and their referents is, in general, not formal and arbitrary, but rather that the relationship is a close linkage between action in the world, language, thought, and meaning (Fauconnier, 1997; Fauconnier & Lakoff, 2009; Fauconnier & Sweetser, 1996; Fauconnier & Turner, 2002; Lakoff, 1987; Lakoff & Johnson, 1980, 1999). For example, if we look back at the opening vignette, our fifth-grade student, Jim, uses a combination of spoken words and gestures to justify his claim that "we can deduce the number of faces if we know the name of the prism." According to a cognitive linguistic frame- work, neither Jim's language nor his bodily actions are unrelated to the way he thinks about the situation. When he refers to the "faces" of the prism, he is using a term that did not develop arbitrarily within the mathematical community, but rather because of its association with the human face, a somewhat planar feature that we present and respond to in the social world. Thus, our physical form serves as a source for naming a mathematical entity, in a nonarbitrary way. [p. 715 starts here]

Similarly, the circular gesture Jim uses to physically encompass all the faces of the prism is related to what cognitive linguistics calls the *image schema* of containment (Johnson, 1987; Lakoff, 1987; Lakoff& Núñez, 2000; Talmy, 2000). Image schemas are "recurrent, stable pat- terns of sensorimotor experience . . . [that] preserve the topological structure of the perceptual whole . . . having internal structures that give rise to constrained inferences" (Johnson, 2007, p. 144). The containment image schema arises from the child's physical experience of filling and emptying containers, experience which builds the notions of "inside," "outside," and "edge" or "bound- ary." This image schema allows Jim to think about the faces of the prism as members of a collection that can be counted, and his circular gesture indicates the boundary of the collection. The containment image schema pro- vides the foundation for many later understandings, both within and outside mathematics, including set member- ship, the domain and range of a function, and bounded regions (Lakoff & Núñez, 2000; Núñez, 2000).

Image schemas can help account for the fact that many mathematical expressions and some symbols evoke space and spatial relationships, even when the subject is not geometry (e.g., "limit," "field," "onto," ⇔). When we talk about "balancing" an equation, or "sup- porting" an argument, from the perspective of cognitive linguistics, this is comprehensible because of our shared experience of balancing and supporting our bodies (as well as building blocks, bicycles, and so on).

An image schema can serve as the source domain for a powerful mechanism in cognitive linguistics, *conceptual metaphor.* Conceptual metaphors are unconscious mappings between two conceptual domains, in which [p. 716 starts here] the inferential structure of the first domain is mapped onto the second (Johnson, 1987; Lakoff, 1987, 1992; Lakoff & Johnson, 1980, 1999). As an example, a common image structure, source-path-goal, is based on our physical experience of traveling from one location (the source) to another (the goal), along a given path (Johnson, 1987). This image schema can be found in multiple areas of mathematics, from addition using the number line (Lakoff & Núñez, 2000) to functions and graphing (Bazzini, 2001; Ferrara, 2003, 2014; Font, Bolite, & Acevedo, 2010) to continuity (Núñez, Edwards, & Matos, 1999). It can even provide the source domain for understanding proof, a concept with no obvious relation- ship to movement through space. Figure 26.6 illustrates the metaphorical mapping from the internal structure of the source-path-goal schema to the explicit form of a mathematical proof (Edwards, 2010).

Empirical support for this metaphor can be found in both the speech and gestures of a doctoral student talking about mathematical proof, shown in Figure 26.7 (Edwards, 2010):

Student: 'cause you start figuring out, I'm starting at **point a and ending up at point b**. There's gonna be **some road//where does it go through**? And can I show that **I can get through there**? (bold indicates speech coordinated with gestures). (Edwards, 2010, p. 333)

The source domain for a conceptual metaphor can be drawn from experience in the physical world (in which case it is called a grounding metaphor), or it can be drawn from an existing conceptualization (creating a linking metaphor, which can yield more abstract concepts— connecting, for example, subdomains within mathematics; Lakoff & Núñez, 2000; Núñez, 2008).

Source Domain: Journey Target Domain: Proof

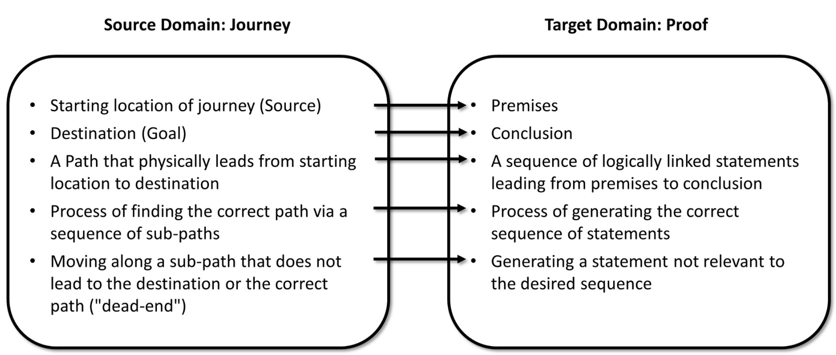


Figure 26.6. *Conceptual metaphor "a proof is a journey." Adapted from "Doctoral Students, Embodied Discourse and Proof" by L. D. Edwards, in M. M. F. Pinto and T. F.*

Kawasaki (Eds.), Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education *(Vol. 2), 2010, Belo Horizonte, Brazil: PME.*

Figure 26.7. *Gesture indicating the source-path-goal image schema. Adapted from "Doctoral Students, Embodied Discourse and Proof" by L. D. Edwards, in M. M. F. Pinto and T. F. Kawasaki (Eds.),* Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education *(Vol. 2), 2010, Belo Horizonte, Brazil: PME.*

An example of conceptual metaphor at a more sophisticated level of mathematics can be found in an analysis of undergraduate students' metaphors for limits. Oehrtman (2009) used interviews and written assignments to ask students in an introductory calculus class about limits. He identified five clusters of metaphors for limits used consistently by his students: "(*a*) a collapse in dimension, (*b*) approximation and error analyses, (*c*) proximity in a space of point-locations, (*d*) a small physical scale beyond which nothing exists, and (*e*) the treatment of infinity as a number" (p. 396). For example, students who were asked about the Taylor series for sin *x* used the "proximity" or "physical closeness" metaphor in statements such as, "the closer the polynomials will *wrap themselves* [emphasis added] around the original function" and "the polynomial becomes more and more *loosely fitted* [emphasis added] around the curve" (p. 417).

The examples above illustrate the use of the theory of conceptual metaphor to analyze specific mathematical topics and concepts; however, there is a more fundamental metaphor underlying mathematical discourse, one which may have contributed to perennial arguments over the ontological status of mathematics. This is the metaphor used when talking about mathematical entities as if they had a physical existence, that is, as objects (Font, Godino, Planas, & Acevedo, 2009; Lakoff & Núñez, 2000). This metaphor is evident when people talk about "manipulating" an equation or ask "how many fives in twenty?" Lakoff and Núñez (2000) spelled out this use of language in what they call the onto- logical metaphor, which takes physical objects as the source domain for conceptualizing mathematical entities; however, this phenomenon was noted earlier by Pimm (1987) and Sfard (1994). Sfard (1994) noted, "The fact that we use the word 'existence' with reference to abstract objects (as in existence theorems) reflects in the most persuasive way the metaphorical nature of the world of abstract ideas" (p. 48). The objectual metaphor, as it is termed by Font and colleagues (Font et al., 2009), offers great advantages—it allows someone carrying out mathematical work to treat symbols as well as abstract ideas as objects, "moving" and "transforming" them, thus radically reducing the cognitive load that would be required if every mathematical sign had to be grounded in its logical mathematical definition.

The construct of conceptual integration emerged within cognitive linguistics at about the same time as conceptual metaphor; however, where metaphor com- prises a unidirectional mapping from exactly one source domain to exactly one target domain, conceptual integration can involve multiple input spaces (Fauconnier & Lakoff, 2009). Conceptual integration, also known as conceptual blending, "connects input spaces, projects selectively to a blended space, and develops emergent structure" (Fauconnier & Turner, 2002, p. 89).

Conceptual integration often builds on existing blends, in a manner similar to linking metaphors (Lakoff & Núñez, 2000). For example, the conceptual mapping for the complex numbers relies on the existence of the blend for the number line as well as the blend for the Cartesian coordinate plane, each of which are blends themselves (Fauconnier & Turner, 2002; Lakoff & Núñez, 2000). The first input space for the "complex numbers" blend consists of the oriented coordinate plane with vector arithmetic. For the second input space, the blend draws on positive and negative real numbers and their associated operations and proper- ties. The blended space yields the complex numbers in the complex plane, in which each element is simultaneously a number and a vector, an emergent quality that [p.717 starts here] was not present in either of the input spaces. The blend creates other new, emergent structures; for example, the blend of the coordinate axes with the positive and negative numbers yields a complex number made up of a real part, displayed on the x-axis, and a complex part on the y-axis. In addition, "running the blend" (working out its entailments through a mechanism called "elaboration") allows addition and multiplication to be redefined to work consistently and coherently in the new space (Fauconnier & Turner, 2002). Fauconnier and Turner (2002) point out that the generic space for this blend (that is, the elements that the two spaces share in common) is made up of two operations with a specific set of properties, namely, associativity, commutativity, identities, inverses, and distributivity of one operation over the other. This combination of operations and proper- ties has come to be seen, and labeled, as a mathematical entity in itself, the commutative ring.

Another example of conceptual integration is drawn from the work of Zandieh, Roh, and Knapp (2014), who analyzed the logical frameworks that a group of students used in working together to create a proof. The task set for the students was to create a proof showing that one conditional statement implied a second conditional statement, specifically, that "either Euclid's Fifth Postulate (EFP) implies Playfair's Parallel Postulate (PPP) or PPP implies EFP" (p. 213). The researchers noted two different conceptual blends for the logical framing of their proof: "a simple proving frame" (which was inadequate for this particular task) and a "conditional implies conditional proving frame" (p. 214). The researchers also analyzed the ways in which the students utilized the visual information associated with each postulate and proposed that the mechanism of conceptual integration allowed the students to merge this information to find the key idea needed for their proof. They also note that conceptual blending does not always lead to correct thinking, offering an example in which three students "condense the premise and conclusion of EFP in a way that loses the implication structure" (p. 228).

The research of Yoon et al. (2011) provides another example of how the analysis of gesture has been integrated into cognitive linguistics. They have investigated what they refer to as "virtual mathematical constructs— constructs that are created via sensuous cognition in a mathematical gesture space through the multimodal use of gestures, speech and other linked semiotic systems" (p. 893). That is, they point out that a student or teacher can utilize the affordances offered by the body, specifically the hand and arm, to establish mathematical meanings through linked gesture and speech (McNeill, 2000, 2005), giving the example of a student using a straight hand held at a (varying) angle to represent the changing gradient of an antiderivative graph. This is possible because elements of the physical world (in this case the hand and arm) can be recruited to serve as one input space for a conceptual blend, a particular type of input called *real space* (Liddell, 1998). The (student's understanding of the) mathematical domain is the second input space. In the blend, the student's flat or angled palm and fingers are mapped to the gradient of the tan- gent line, and the hand's motion and location to movement along the antiderivative graph (Yoon et al., 2011). Via conceptual integration, physical action becomes an important resource for the students in constructing an understanding of the mathematical content.

The field of cognitive linguistics offers a powerful theoretical framework and a set of productive tools for understanding mathematics that can be applied equally well to a child's earliest construction of number sense or a mathematician's elaboration of abstract structures. The theory of cognitive linguistics is conceptually coherent, supported empirically via multiple methodologies, and connects with other advances in cognitive science, including neurological research (e.g., Fields, 2013; Guhe et al., 2011; Winter, Marghetis, & Matlock, 2015). One of the principles of embodied cognition is that of cognitive continuity (Johnson, 2007), under which mathematics is not ontologically different from other realms of cognition and action. Instead, the cognitive mechanisms that have allowed humans to survive and thrive over millennia have also supported the development of mathematical thought and other conceptual domains.

Looking to the Future: New Problems, Tensions, and Questions

In this chapter we presented an overview of embodiment in mathematics education. The starting point is a general claim that contemporary embodiment theories make about meaning and cognition: that meaning and cognition are deeply rooted in a physical, material, embodied existence. For instance, Sheets-Johnstone (2009) argues that, as a result of our biological makeup, we are naturally equipped with a range of archetypal corporeal- kinetic forms and relations that constitute the basis on which we make our ways into the world (see also Roth, 2012; Seitz, 2000). Based on this general claim about the embodied nature of meaning and cognition, embodied theories attempt to provide answers to questions of how meaning arises and of how thought is related to action, emotion, and perception (Edwards, 2011). Since there is not just one way in which to theorize the cognitive [p. 118 starts here] role of the body, it is not surprising to find a variety of contemporary perspectives on embodied cognition (for an overview see Wilson, 2002; for a discussion see Radford, 2013). Their differences rest, among others, on conceptions about the senses, matter, cognition, and the body itself. The differences can be tracked back to the long-standing philosophical problem concerning the relationship of the body, the senses, and the mind. The first part of the chapter dealt indeed with a brief account of embodiment in the main epistemological Western traditions, namely the rationalist and the empiricist traditions. The first part was an attempt to show the legendary painful struggle of Western thought to understand the epistemological question of the body to better understand the historical background from which contemporary embodiment theories emerge.

The theoretical perspectives that contemporary embodiment theories bring forward open up new possibilities to envision teaching and learning in new ways. In the second part of the chapter we referred to some approaches that have been incorporated and developed within mathematics education. Our overview, although limited, nonetheless shows the variety of approaches and some of their differences (e.g., Tall's and Dubinsky's "process-object" theories, Roth's radical phenomenological approach, Nemirovsky and collaborators' Husserlian phenomenological approach, de Freitas and Sinclair's inclusive materialist approach, Arzarello and collaborators' Vygotskian semiotic approach). In the rest of the chapter we developed with more detail two other approaches—the "sensuous cognition" approach embedded in cultural historical activity theory and inspired by a neo-Hegelian dialectic materialism and the cognitive linguistics approach inspired by the work of Lakoff (1987, 1992) and Núñez (2000), among others. These approaches emphasize the embodied nature of cognition and provide us with an opportunity to see thematic and conceptual differences in the manner in which the body, gestures, the senses, language, and artifacts are considered. As a result of these thematic and conceptual differences, the corresponding research questions and methodologies vary.

Embodiment in mathematics education is still an emerging and developing research area. There is considerable work to be done at the conceptual and methodological levels. From the viewpoint of the "sensuous cognition" approach, we need to better understand the development of students' cognition (which includes thinking, volition, and emotion) not as a strict mental event but rather as a tangible phenomenon that is simultaneously conceptual, embodied, *and* material. That is, we need to examine cognition as a phenomenon that arises from and brings together, in the Hegelian dialectical sense, the sentient subjects and cultural forms of thought through language, body, artifacts, and semiotic activity more generally. We need, for instance, to better understand the social-and-individual activity-bound dialectic relationship between (inner and outer) speech, sensuous forms of imagination, gestures, tactility, and actions with cultural artifacts (including mathematical symbolism).

From the perspective of cognitive linguistics, the out- line of a comprehensive framework linking bodily experience with mathematical knowledge and practices is emerging. This framework offers specific tools for the analysis of the wide range of modalities found in mathematical actions and expression, from speech to symbols to images to gestures (Edwards & Robutti, 2014). Yet many questions remain. In particular, it is not enough to analyze mathematical ideas "after the fact"; instead, we need to learn more about how best to employ various modalities in teaching mathematics. In addition, the physiological and neural mechanisms involved in linking physical experience to mathematical thought remain relatively unexplored. For example, it would be interesting to compare the neurological correlates of learning arithmetic via specific hands-on manipulatives versus others, or versus rote memorization. Cognitive linguistic theory would predict differences in the resulting conceptual metaphors for arithmetic; if found, would such differences also be reflected in neurological structure or function?

The current increasing interest in embodiment in mathematics education offers hope that this research field will continue to attract more researchers who will continue, expand, and envision new theoretical and practical paths to improving the teaching and learning of mathematics.

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